Some comments on rigorous quantum field path integrals in the analytical regularization scheme

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Abstract

Through the systematic use of the Minlos theorem on the support of cylindrical measures on R^{∞}, we produce several mathematically rigorous path integrals in interacting euclidean quantum fields with Gaussian free measures defined by generalized powers of the Laplacean operator, defined on finite volume space-times.

1 Introduction

Since the result of R.P. Feynman on representing the initial value solution of Schrodinger Equation by means of an analytically time continued integration on a infinite - dimensional space of functions, the subject of Euclidean Functional Integrals representations for Quantum Systems has became the mathematical - operational framework to analyze Quantum Phenomena and stochastic systems as showed in the previous decades of research on Theoretical Physics ([1]–[3]).

One of the most important open problem in the mathematical theory of Euclidean Functional Integrals is that related to implementation of sound mathematical approximations to these Infinite-Dimensional Integrals by means of Finite-Dimensional approximations outside of the always used [computer oriented] Space-Time Lattice approximations (see [2], [3] - chap.

9). As a first step to tackle upon the above cited problem it will be needed to characterize mathematically the Functional Domain where these Functional Integrals are defined.

The purpose of this note is to present the formulation of Euclidean Quantum Field theories as Functional Fourier Transforms by means of the Bochner-Martin-Kolmogorov theorem for Topological Vector Spaces ([4], [5] - theorem 4.35) and suitable to define and analyze rigorously Functional Integrals by means of the well-known Minlos theorem ([5] - theorem 4.312 and [6] - part 2) and presented in full in Appendix 1.

We thus present news results on the difficult problem of defining rigorously infinite-dimensional quantum field path integrals in general finite volume space times $\Omega \subset R^\nu \quad (\nu = 2, 4, \dots)$ by means of the analytical regularization scheme.

2 Some rigorous quantum field path integral in the Analytical regularization scheme

Let us thus start our analysis by considering the Gaussian measure associated to the (infrared regularized) α-power ($\alpha > 1$) of the Laplacean acting on $L^2(R^2)$ as an operatorial quadratic form

$$\begin{aligned} Z^{(0)}_{\alpha,\varepsilon_{IR}}[j] &= \exp\left\{ -\frac{1}{2} \left\langle j, (-\Delta)^{-\alpha}_{\varepsilon_{IR}} \right\rangle_{L^2(R^2)} \right\}^{(*)} \\ &= \int d^{(0)}_{\alpha,\varepsilon_{IR}}\mu\,[\varphi]\ \exp\left(i \left\langle j, \varphi \right\rangle_{L^2(R^2)} \right) \end{aligned} \tag{1-a}$$

Here $\varepsilon_{IR} > 0$ denotes the infrared cut off.

It is worth call the reader attention that due to the infrared regularization introduced on eq (1-a), the domain of the Gaussian measure ([4]–[6]) is given by the space of square integrable

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$^{(*)}$ Rigorously:

$$(-\Delta)^{-\alpha}_{\varepsilon_{IR}} = \frac{I_\Omega(x)(2\pi)^{n/2}}{\mathrm{vol}(\Omega)} [(-\Delta)^\alpha + \varepsilon_{IR}]^{-1},$$

here $I_\Omega(x) = \begin{cases} 1 & x \in \Omega \in R^n \\ 0 & x \notin \Omega \end{cases}$

functions on R^2 by the Minlos theorem of Appendix 1, since for $\alpha > 1$, the operator $(-\Delta)^{-\alpha}_{\varepsilon_{IR}}$ defines a classe trace operator on $L^2(R^2)$, namely

$$Tr_{\oint_1}((-\Delta)^{-\alpha}_{\varepsilon_{IR}}) = \int d^2k \frac{1}{(|K|^{2\alpha} + \varepsilon_{IR})} < \infty \tag{1-b}$$

This is the only point of our analysis where it is needed to consider the infra-red cut off considered on the spectral resolution eq (1-a). As a consequence of the above remarks, one can analize the ultra-violet renormalization program in the following interacting model proposed by us and defined by an interaction $g_{\text{bare}} V(\varphi(x))$, with $V(x)$ being the Fourier Transformed of a compact support essentially bounded measurable or square integrable function. Note that $V(x)$ is thus an entire function with an exponential bound by the Wiener theorem (see Theorem 19.3 [11]). Note that

$$\left| \int_\Omega \left[\int_{-\Lambda}^{\Lambda} e^{ik\varphi(x)} \tilde{V}(k) dk \right] d^2x \right| \leq 2\Lambda \cdot \text{vol}(\Omega) ||\tilde{V}||_{L^\infty} < \infty,$$

for any finite volume region $\Omega \subset R^3$, where $\text{Supp}\, \tilde{V}(k) \subset [-\Lambda, \Lambda]$.

Let us show that by defining a renormalized coupling constant as (with $g_{ren} < 1$) with a finite volume Ω cut off built in wich will be removed at the renormalized limit (see eq.(6)–eq.(9) below).

$$g_{\text{bare}} = \frac{g_{\text{ren}}}{(1-\alpha)^{1/2}} \tag{2}$$

one can show that the interaction function

$$\exp\left\{ -g_{\text{bare}}(\alpha) \int_\Omega d^2x\, V(\varphi(x)) \right\} \tag{3}$$

is an integrable function on $L^1(L^2(R^2), d^{(0)}_{\alpha,\varepsilon_{IR}}\mu\,[\varphi])$ and leads to a well-defined ultra-violet path integral in the limit of $\alpha \to 1$.

The proof is based on the following estimates.

Since almost everywhere we have the pointwise limit

$$\begin{aligned} &\exp\left\{ -g_{\text{bare}}(\alpha) \int d^2x\, V(\varphi(x)) \right\} \\ &\lim_{N\to\infty} \left\{ \sum_{n=0}^{N} \frac{(-1)^n (g_{\text{bare}}(\alpha))^n}{n!} \int_{[-\Lambda,\Lambda]} dk_1 \cdots dk_n\, \tilde{V}(k_1) \cdots \tilde{V}(k_n) \right. \\ &\quad \left. \times \int_\Omega dx_1 \cdots dx_n\, e^{ik_1\varphi(x_1)} \cdots e^{ik_n\varphi(x_n)} \right\} \end{aligned} \tag{4}$$

we have that the upper-bound estimate below holds true

$$\Big|Z^{\alpha}_{\varepsilon_{IR}}[g_{\text{bare}}]\Big| \leq \Big|\sum_{n=0}^{\infty}\frac{(-1)^n(g_{\text{bare}}(\alpha))^n}{n!}\int_{[-\Lambda,\Lambda]} dk_1\cdots dk_n\,\tilde{V}(k_1)\cdots\tilde{V}(k_n)$$
$$\int_{\Omega}dx_1\cdots dx_n\int d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi](e^{i\sum_{\ell=1}^{N}k_\ell\varphi(x_\ell)})\Big| \tag{5-a}$$

with

$$Z^{\alpha}_{\varepsilon_{IR}}[g_{\text{bare}}] = \int d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi]\,\exp\left\{-g_{\text{bare}}(\alpha)\int_{\Omega}d^2x\,V(\varphi(x))\right\} \tag{5-b}$$

we have, thus, the more suitable form after realizing the d^2k_i and $d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi]$ integrals respectivelly[(**)]

$$\Big|Z^{\alpha}_{\varepsilon_{IR}=0}[g_{bare}]\Big| \leq \sum_{n=0}^{\infty}\frac{(g_{bare}(\alpha))^n}{n!}\Big(||\tilde{V}||_{L^{\infty}(R)}\Big)^n$$
$$\Big|\int_{R^2}dx_1\cdots dx_n\,\det{}^{-\frac{1}{2}}\Big[G^{(N)}_{\alpha}(x_i,x_j)\Big]_{\substack{1\leq i\leq N\\ 1\leq j\leq N}}\Big| \tag{6}$$

Here $[G^{(N)}_{\alpha}(x_i,x_j)]_{\substack{1\leq i\leq N\\ 1\leq j\leq N}}$ denotes the $N\times N$ symmetric matrix with the (i,j) entry given by the positive Green-function of the α-Laplacean (without the infra-red cut off here!).

$$G_{\alpha}(x_i,x_j) = |x_i-x_j|^{2(\alpha-1)}\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}\frac{1}{2\pi}2^{2(1-\alpha)} \tag{7}$$

At this point, we call the reader attention that we have the formulae on the asymptotic behavior for $\alpha\to 1$ and $\alpha<1$.

$$\left\{\lim_{\substack{\alpha\to 1\\ \alpha>1}}\det{}^{-\frac{1}{2}}[G^{(N)}_{\alpha}(x_i,x_j)]\right\}\sim(1-\alpha)^{N/2}\times\left(\Big|\frac{(N-1)(-1)^N}{\pi^{N/2}}\Big|\right)^{-\frac{1}{2}} \tag{8}$$

After substituting eq.(8) into eq.(6) and taking into account the hypothesis of the compact support of the nonlinearity $\tilde{V}(k)$, one obtains the finite bound for any value $g_{rem}>0$, without

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(**) Note that:

$$\int_{\Omega}d^2x_1\ldots d^2x_n e^{-\frac{1}{2}\sum_{i,j}^{N}k_ik_j\overbrace{G_{\alpha}(x_i,x_j)}^{\geq 0}} \leq \int_{R^2}d^2x_1\ldots d^2x_n e^{-\frac{1}{2}\sum_{i,j}^{N}k_ik_j\overbrace{G_{\alpha}(x_i,x_j)}^{\geq 0}}$$

the finite volume cut off and producing a proof for the convergence of the perturbative expansion in terms of the renormalized coupling constant for the model already defined in infinite volume

$$\lim_{\alpha\to 1}\left|Z^{\alpha}_{\varepsilon_{IR}=0}[g_{bare}(\alpha)]\right| \leq \sum_{n=0}^{\infty}\frac{(\|\tilde{V}\|_{L^\infty(R)})^n}{n!}\left(\frac{g_{ren}}{(1-\alpha)^{\frac{1}{2}}}\right)^n \times \frac{(1^n)}{\sqrt{n}}(1-\alpha)^{n/2}$$
$$\leq e^{\sqrt{\pi}g_{ren}\|\tilde{V}\|_{L^\infty(R)}} < \infty \tag{9}$$

Another important rigorously defined functional integral is to consider the following α-power Klein Gordon operator on Euclidean space-time

$$\mathcal{L}_{\Omega}^{-1} = \left(\frac{I_{\Omega}(x)(2\pi)^{\nu/2}}{\mathrm{vol}(\Omega)}\right)[(-\Delta)^{\alpha}+m^2]^{-1} \tag{10}$$

with m^2 a positive "mass" parameters.

Let us note that $\mathcal{L}^{-1}$ is an operator of class trace on $L^2(R^\nu)$ if and only if the result below holds true

$$Tr_{L^2(R^\nu)}(\mathcal{L}^{-1}) = \int d^\nu k \frac{1}{k^{2\alpha}+m^2} = \bar{C}(\nu)\, m^{(\frac{\nu}{\alpha}-2)} \times \left\{\frac{\pi}{2\alpha}\mathrm{cosec}\frac{\nu\pi}{2\alpha}\right\} < \infty \tag{11}$$

namely if

$$\alpha > \frac{\nu}{2} \tag{12}$$

In this case, let us consider the double functional integral with functional domain $L^2(R^\nu)$

$$Z[j,k] = \int d_G^{(0)}\,\beta[v(x)]$$
$$\times \int d^{(0)}_{\mathcal{L}\Omega}\mu[\varphi]$$
$$\times \exp\left\{i\int d^\nu x\,(j(x)\,\varphi(x)+k(x)\,v(x))\right\} \tag{13}$$

where the Gaussian functional integral on the fields $V(x)$ has a Gaussian generating functional defined by a $\oint_1$-integral operator with a positive defined kernel $g(|x-y|)$, namely

$$Z^{(0)}[k] = \int d_G^{(0)}\beta[v(x)]\,\exp\left\{i\int d^\nu x\,k(x)v(x)\right\}$$
$$= \exp\left\{-\frac{1}{2}\int_{\Omega} d^\nu x\int_{\Omega} d^\nu y\,(k(x)\,g(|x-y|)\,k(y))\right\} \tag{14}$$

By a simple direct application of the Fubbini-Tonelli theorem on the exchange of the integration order on eq.(13), lead us to the effective $\lambda\varphi^4$ - like well-defined functional integral representation

$$
\begin{aligned}
Z_{\text{eff}}[j] = \int & d_{\mathcal{L}}^{(0)}\mu[\varphi][\varphi(x)] \\
& \exp\left\{-\frac{1}{2}\int_\Omega d^\nu x d^\nu y \,|\varphi(x)|^2\, g(|x-y|)\,|\varphi(y)|^2\right\} \\
& \times \exp\left\{i\int_\Omega d^\nu x\, j(x)\varphi(x)\right\}
\end{aligned}
\tag{15}
$$

Note that if one introduces from the begining a bare mass parameters m^2_{bare} depending on the parameters α, but such that it always satisfies eq.(11) one should obtains again eq.(15) as a well-defined measure on $L^2(R^\nu)$. Of course that the usual pure Laplacean limit of $\alpha \to 1$ on eq.(10), will needed a renormalization of this mass parameters ($\lim\limits_{\alpha\to 1} m^2_{bare}(\alpha) = +\infty$!) as much as done in the previous example.

Let us continue our examples by showing again the usefulness of the precise determination of the functional - distributional structure of the domain of the functional integrals in order to construct rigorously these path integrals without complicated limit procedures.

Let us consider a general R^ν Gaussian measure defined by the Generating functional on $S(R^\nu)$ defined by the α-power of the Laplacean operator $-\Delta$ acting on $S(R^\nu)$ with a of small infrared regularization mass parameter μ^2 as in eq.(1-a)

$$
\begin{aligned}
Z_{(0)}[j] &= \exp\left\{-\frac{1}{2}\Big\langle j,\, ((-\Delta)^\alpha + \mu_0^2)^{-1}\, j\Big\rangle_{L^2(R^\nu)}\right\} \\
&= \int_{E^{alg}(S(R^\nu))} d_\alpha^{(0)}\,\mu[\varphi]\,\exp(i\,\varphi(j))
\end{aligned}
\tag{16}
$$

An explicitly expression in momentum space for the Green function of the α-power of $(-\Delta)^\alpha + \mu_0^2$ given by

$$
((-\Delta)^{+\alpha} + \mu_0^2)^{-1}(x-y) = \int \frac{d^\nu k}{(2\pi)^\nu}\, e^{ik(x-y)} \left(\frac{1}{k^{2\alpha}+\mu_0^2}\right) \tag{17}
$$

Here $\bar{C}(\nu)$ is a ν-dependent (finite for ν-values !) normalization factor.

Let us suppose that there is a range of α-power values that can be choosen in such way that

one satisfies the constraint below

$$\int_{E^{alg}(S(R^\nu))} d_\alpha^{(0)}\,\mu[\varphi](\|\varphi\|_{L^{2j}(R^\nu)})^{2j} < \infty \tag{18}$$

with $j = 1, 2, \cdots, N$ and for a given fixed integer N, the highest power of our polinomial field interaction. Or equivalently, after realizing the φ-Gaussian functional integration, with a space-time cutt off volume Ω on the interaction to be analyzed on eq.(16)

$$\begin{aligned}\int_\Omega d^\nu x[(-\Delta)^\alpha + \mu_0^2]^{-j}(x,x) &= \mathrm{vol}(\Omega) \times \left(\int \frac{d^\nu k}{k^{2\alpha}+\mu_0^2}\right)^j \\ &= C_\nu(\mu_0)^{(\frac{\nu}{\alpha}-2)} \times \left(\frac{\pi}{2\alpha}\mathrm{cosec}\frac{\nu\pi}{2\alpha}\right) < \infty\end{aligned} \tag{19}$$

For $\alpha > \frac{\nu}{2}$, one can see by the Minlos theorem that the measure support of the Gaussian measure eq.(16) will be given by the intersection Banach space of measurable Lebesgue functions on R^ν instead of the previous one $E^{alg}(S(R^\nu))$ ([4]–[6]).

$$\mathcal{L}_{2N}(R^\nu) = \bigcap_{j=1}^{N} (L^{2j}(R^\nu)) \tag{20}$$

In this case, one obtains that the finite - volume $p(\varphi)_2$ interactions

$$\exp\left\{-\sum_{j=1}^{N} \lambda_{2j} \int_\Omega (\varphi^2(x))^j\, dx\right\} \le 1 \tag{21}$$

is mathematically well-defined as the usual pointwise product of measurable functions and for positive coupling constant values $\lambda_{2j} \ge 0$. As a consequence, we have a measurable functional on $L^1(\mathcal{L}_{2N}(R^\nu);\, d_\alpha^{(0)}\,\mu[\varphi])$ (since it is bounded by the function 1). So, it makes sense to consider mathematically the well-defined path - integral on the full space R^ν with those values of the power α satisfying the contraint eq.(17).

$$Z[j] = \int_{\mathcal{L}_{2N}(R^\nu)} d_\alpha^{(0)}\,\mu[\varphi]\, \exp\left\{-\sum_{j=1}^{N} \lambda_{2j} \int_\Omega \varphi^{2j}(x)dx\right\} \times \exp(i\int_{R^\nu} j(x)\varphi(x)) \tag{22}$$

Finally, let us consider a interacting field theory in a compact space-time $\Omega \subset R^\nu$ defined by an iteger even power $2n$ of the Laplacean operator with Dirichlet Boundary conditions as

the free Gaussian kinetic action, namely

$$Z^{(0)}[j] = \exp\left\{-\frac{1}{2}\Big\langle j, (-\Delta)^{-2n} j\Big\rangle_{L^2(\Omega)}\right\}$$
$$= \int_{W_2^n(\Omega)} d^{(0)}_{(2n)}\,\mu[\varphi]\,\exp(i\langle j,\varphi\rangle_{L^2(\Omega)}) \tag{23}$$

here $\varphi \in W_2^n(\Omega)$ - the Sobolev space of order n which is the functional domain of the cylindrical Fourier Transform measure of the Generating functional $Z^{(0)}[j]$, a continuous bilinear positive form on $W_2^{-n}(\Omega)$ (the topological dual of $W_2^n(\Omega)$) ([4]–[6]).

By a straightforward application of the well-known Sobolev immersion theorem, we have that for the case of

$$n - k > \frac{\nu}{2} \tag{24}$$

including k a real number the functional Sobolev space $W_2^n(\Omega)$ is contained in the continuously fractional differentiable space of functions $C^k(\Omega)$. As a consequence, the domain of the Bosonic functional integral can be further reduced to $C^k(\Omega)$ in the situation of eq.(24)

$$Z^{(0)}[j] = \int_{C^k(\Omega)} d^{(0)}_{(2n)}\,\mu[\varphi]\,\exp(i\langle j,\varphi\rangle_{L^2(\Omega)}) \tag{25}$$

That is our new result generalizing the Wiener theorem on Brownian paths in the case of $n = 1$, $k = \frac{1}{2}$ and $\nu = 1$

Since the bosonic functional domain on eq.(25) is formed by real functions and not distributions, we can see straightforwardly that any interaction of the form

$$\exp\left\{-g\int_\Omega F(\varphi(x))d^\nu x\right\} \tag{26}$$

with the non-linearity $F(x)$ denoting a lower bounded real function ($\gamma > 0$)

$$F(x) \geq -\gamma \tag{27}$$

is well-defined and is integrable function on the functional space $(C^k(\Omega),\ d^{(0)}_{(2n)}\,\mu[\varphi])$ by a direct application of the Lebesque theorem

$$\left|\exp\left\{-g\int_\Omega F(\varphi(x))\,d^\nu x\right\}\right| \leq \exp\{+g\gamma\} \tag{28}$$

At this point we make a subtle mathematical remark that the infinite volume limit of eq.(25) - eq.(26) is very difficult, since one looses the Garding - Poincaré inequalite at this limit for those elliptic operators and, thus, the very important Sobolev theorem. The probable correct procedure to consider the thermodynamic limit in our Bosonic path integrals is to consider solely a volume cut off on the interaction term Gaussian action as in eq.(22) and there search for $\mathrm{vol}(\Omega) \to \infty$ ([7]–[10]).

As a last remark related to eq.(23) one can see that a kind of "fishnet" exponential generating functional

$$Z^{(0)}[j] = \exp\left\{ -\frac{1}{2} \Big\langle j, \exp\{-\alpha\Delta\} j \Big\rangle_{L^2(\Omega)} \right\} \tag{29}$$

has a Fourier transformed functional integral representation defined on the space of the infinitelly differentiable functions $C^\infty(\Omega)$, which physically means that all field configurations making the domain of such path integral has a strong behavior like purely nice smooth classical field configurations.

As a last important point of this note, we present an important result on the geometrical characterization of massive free field on an Euclidean Space-Time ([10]).

Firstly we announcing a slightly improved version of the usual Minlos Theorem ([4]).

Theorem 3. Let E be a nuclear space of tests functions and $d\mu$ a given σ-measure on its topologic dual with the strong topology. Let $\langle\, , \,\rangle_0$ be an inner product in E, inducing a Hilbertian structure on $\mathcal{H}_0 = \overline{(E, \langle\, , \,\rangle_0)}$, after its topological completation.

We suppose the following:

a) There is a continuous positive definite functional in $\mathcal{H}_0$, $Z(j)$, with an associated cylindrical measure $d\mu$.

b) There is a Hilbert-Schmidt operator $T \colon \mathcal{H}_0 \to \mathcal{H}_0$; invertible, such that $E \subset$ Range (T), $T^{-1}(E)$ is dense in $\mathcal{H}_0$ and $T^{-1} \colon \mathcal{H}_0 \to \mathcal{H}_0$ is continuous.

We have thus, that the support of the measure satisfies the relationship

$$\text{support } d\mu \subseteq (T^{-1})^*(\mathcal{H}_0) \subset E^* \tag{30}$$

At this point we give a non-trivial application of ours of the above cited Theorem 3.

Let us consider an differential inversible operator $\mathcal{L}\colon S'(R^N) \to S(R)$, together with an positive inversible self-adjoint elliptic operator $P\colon D(P) \subset L^2(R^N) \to L^2(R^N)$. Let H_α be the following Hilbert space

$$H_\alpha = \left\{ \overline{S(R^N), \langle P^\alpha \varphi, P^\alpha \varphi \rangle_{L^2(R^N)} = \langle \, , \rangle_\alpha} \,,\ \text{for } \alpha \text{ a real number} \right\}. \tag{31}$$

We can see that for $\alpha > 0$, the operators below

$$\begin{aligned} P^{-\alpha}\colon L^2(R^N) &\to \mathcal{H}_{+\alpha} \\ \varphi &\to (P^{-\alpha}\varphi) \end{aligned} \tag{32}$$

$$\begin{aligned} P^{\alpha}\colon \mathcal{H}_{+\alpha} &\to L^2(R^N) \\ \varphi &\to (P^{\alpha}\varphi) \end{aligned} \tag{33}$$

are isometries among the following sub-spaces

$$\overline{D(P^{-\alpha}), \langle \, , \rangle_{L^2})} \text{ and } H_{+\alpha}$$

since

$$\langle P^{-\alpha}\varphi, P^{-\alpha}\varphi \rangle_{\mathcal{H}_{+\alpha}} = \langle P^\alpha P^{-\alpha}\varphi, P^\alpha P^{-\alpha}\varphi \rangle_{L^2(R^N)} = \langle \varphi, \varphi \rangle_{L^2(R^N)} \tag{34}$$

and

$$\langle P^\alpha f, P^\alpha f \rangle_{L^2(R^N)} = \langle f, f \rangle_{H_{+\alpha}} \tag{35}$$

If one considers T a given Hilbert-Schmidt operator on H_α, the composite operator $T_0 = P^\alpha T P^{-\alpha}$ is an operator with domain being $D(P^{-\alpha})$ and its image being the Range (P^α). T_0 is clearly an invertible operator and $S(R^N) \subset$ Range (T) means that the equation $(TP^{-\alpha})(\varphi) = f$ has always a non-zero solution in $D(P^{-\alpha})$ for any given $f \in S(R^N)$. Note that the condition that $T^{-1}(f)$ be a dense subset on Range $(P^{-\alpha})$ means that

$$\langle T^{-1} f, P^{-\alpha}\varphi \rangle_{L^2(R^N)} = 0 \tag{36}$$

has as unique solution the trivial solution $f \equiv 0$.

Let us suppose too that $T^{-1}\colon S(R^N) \to H_\alpha$ be a continuous application and the bilinear term $(\mathcal{L}^{-1}(j))(j)$ be a continuous application in the Hilbert spaces $H_{+\alpha} \supset S(R^N)$, namely: if $j_n \xrightarrow{L^2} j$, then $\mathcal{L}^{-1}\colon P^{-\alpha} j_n \xrightarrow{L^2} \mathcal{L}^{-1} P^{-\alpha} j$, for $\{j_n\}_{n \in \mathbb{Z}}$ and $j_n \in S(R^N)$.

By a direct application of the Minlos Theorem, we have the result

$$Z(j) = \exp\left\{-\frac{1}{2}[\mathcal{L}^{-1}(j)(j)]\right\} = \int_{(T^{-1})^*\,H_\alpha} d\mu(T)\exp(iT(j)) \tag{37}$$

Here the topological space support is given by

$$\begin{aligned}(T^{-1})^*\mathcal{H}_\alpha &= \left[\left(P^{-\alpha}T_0P^\alpha\right)^{-1}\right]^*\left(\overline{(P^\alpha(S(R^N)))}\right)\\ &= \left[(P^\alpha)^*(T_0^{-1})^*(P^{-\alpha})^*\right]P^\alpha(S(R^N))\\ &= P^\alpha T_0^{-1}(L^2(R^N))\end{aligned} \tag{38}$$

In the important case of $\mathcal{L} = (-\Delta + m^2)\colon S'(R^N) \to S(R^N)$ and $T_0T_0^* = (-\Delta+m^2)^{-2\beta} \in \oint_1(L^2(R^N))^{(***)}$ since $Tr(T_0T_0^*) = \dfrac{1}{2(m^2)^\beta}\left(\dfrac{m^2}{1}\right)^{\frac{N}{2}}\dfrac{\Gamma(\frac{N}{2})\Gamma(2\beta-\frac{N}{2})}{\Gamma(\beta)} < \infty$ for $\beta > \dfrac{N}{4}$ with the choice $P = (-\Delta+m^2)$, we can see that the support of the measure in the path-integral representation of the Euclidean measure field in R^N may be taken as the measurable sub-set below

$$\text{supp}\ \{d_{(-\Delta+m^2)}\,u(\varphi)\} = (-\Delta+m^2)^{+\alpha}I_\Omega(x)(-\Delta+m^2)^{+\beta}(L^2(R^N)) \tag{39}$$

since $\mathcal{L}^{-1}P^{-\alpha} = (-\Delta+m^2)^{-1-\alpha}$ is always a bounded operator in $L^2(R^N)$ for $\alpha > -1$.

As a consequence each field configuration can be considered as a kind of "fractional distributional" derivative of a square integrable function as written below of the formal infinite volume $\Omega \to R^N$.

$$\varphi(x) = \left[\big(-\Delta+m^2\big)^{\frac{N}{4}+\varepsilon-1}f\right](x) \tag{40}$$

with a function $f(x) \in L^2(R^N)$ and any given $\varepsilon > 0$, even if originally all fields configurations entering into the path-integral were elements of the Schwartz Tempered Distribution Spaces $S'(R^N)$ certainly very "rough" mathematical objects to characterize from a rigorous geometrical point of view.

We have, thus, make a further reduction of the functional domain of the free massive Euclidean scalar field of $S'(R^N)$ to the measurable sub-set as given by eq.(130) denoted by

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(***)

$$T_0T_0^* = \frac{I_\Omega(x)(2\pi)^{N/2}}{\text{vol}(\Omega)}(-\Delta+m^2)^{-2\beta}$$

$W(R^N)$

$$\exp\left\{-\frac{1}{2}\left[(-\Delta+m^2)^{-1}j\right](j)\right\} = \int_{S'(R^N)} d_{(-\Delta+m^2)}\mu(\varphi)\, e^{i\,\varphi(j)}$$
$$= \int_{W(R^N)\subset S'(R^N)} d_{(-\Delta+m^2)}\tilde{\mu}(f)\, e^{i\langle f,(-\Delta+m^2)^{\frac{N}{4}+\varepsilon-1}f\rangle_{L^2(R^N)}} \qquad (41)$$

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APPENDIX

Some Comments on the Support of Functional Measures in Hilbert Space

Let us comment further on the application of the Minlos Theorem in Hilbert Spaces. In this case one has a very simple proof which holds true in general Banach Spaces $(E, ||\quad||)$.

Let us thus, give a cylindrical measures $d^\infty\mu(x)$ in the algebraic dual E^{alg} of a given Banach Space E ([4]–[6]).

Let us suppose either that the function $||x||$ belongs to $L^1(E^{\mathrm{alg}}, d^\infty\mu(x))$. Then the support of this cylindrical measures will be the Banach Space E.

The proof is the following:

Let A be a sub-set of the vectorial space E^{alg} (with the topology of pontual convergence), such that $A \subset E^c$ (so $||x| = +\infty$) (E can always be imbed as a cylindrical measurable sub-set of E^{alg} - just use a Hammel vectorial basis to see that). Let be the sets $A_n = \{x \in E^{\mathrm{alg}} \mid ||x|| \geq n\}$. Then we have the set inclusion $A \subset \bigcap_{n=0}^{\infty} A_n$, so its measure satisfies the estimates below:

$$\begin{aligned}
\mu(A) &\leq \liminf_n \mu(A_n) \\
&= \liminf_n \mu\{x \in E^{\mathrm{alg}} \mid ||x|| \geq n\} \\
&\leq \liminf_n \left\{ \frac{1}{n} \int_{E^{\mathrm{alg}}} ||x|| d^\infty\mu(x) \right\} \\
&= \liminf_n \frac{||x||_{L^1(E^{\mathrm{alg}}, d^\infty_\mu)}}{n} = 0. \qquad (1)
\end{aligned}$$

Leading us to the Minlos theorem that the support of the cylindrical measure in E^{alg} is reduced to the own Banach Space E.

Note that by the Minkowisky inequality for general integrals, we have that $||x||^2 \in L^1(E^{\mathrm{alg}}, d^\infty\mu(x))$. Now it is elementary evaluation to see that if $A^{-1} \in \oint_1(\mathcal{M})$, when $E = \mathcal{M}$, a given Hilbert Space, we have that

$$\int_{\mathcal{M}^{\mathrm{alg}}} d^\infty_A \mu(x) \cdot ||x||^2 = Tr_{\mathcal{M}}(A^{-1}) < \infty. \qquad (2)$$

This result produces another criterium for supp $d_A^{\infty}\mu = \mathcal{M}$ (the Minlos Theorem), when $E = \mathcal{M}$ is a Hilbert Space.

It is easy too to see that if

$$\int_{\mathcal{M}} ||x|| d^{\infty}\mu(x) < \infty \tag{3}$$

then the Fourier-Transformed functional

$$Z(j) = \int_{\mathcal{M}} e^{i(j,x)_{\mathcal{M}}} d^{\infty}\mu(x) \tag{4}$$

is continuous in the norm topology of $\mathcal{M}$.

Otherwise, if $Z(j)$ is not continuous in the origin $0 \in \mathcal{M}$ (without loss of generality), then there is a sequence $\{j_n\} \in \mathcal{M}$ and $\delta > 0$, such that $||j_n|| \to 0$ with

$$\begin{aligned} \delta \le |Z(j_n) - 1| &\le \int_{\mathcal{M}} |e^{i(j_n,x)_{\mathcal{M}}} - 1| d^{\infty}\mu(x) \\ &\le \int_{\mathcal{M}} |(j_n, x)| d^{\infty}\mu(x) \\ &\le ||j_n|| \Big(\int_{\mathcal{M}} ||x|| d^{\infty}\mu(x) \Big) \to 0, \end{aligned} \tag{5}$$

a contradiction with $\delta > 0$.

Finally, let us consider an elliptic operator B (with inverse) from the Sobelev space $\mathcal{M}^{-2m}(\Omega)$ to $\mathcal{M}^{2m}(\Omega)$. Then by the criterium given by eq.(2) if

$$Tr_{L^2(\Omega)}[(I+\Delta)^{+\frac{m}{2}} B^{-1} (I+\Delta)^{+\frac{m}{2}}] < \infty, \tag{6}$$

we will have that the path integral below written is well-defined for $x \in \mathcal{M}^{+2m}(\Omega)$ and $j \in \mathcal{M}^{-2m}(\Omega)$. Namely

$$\exp(-\frac{1}{2}(j, B^{-1}j)_{L^2(\Omega)}) = \int_{\mathcal{M}^{+2m}(\Omega)} d_B\mu(x) \exp(i(j,x)_{L^2(\Omega)}). \tag{7}$$

By the Sobolev theorem which means that the embeeded below is continuous (with $\Omega \subseteq R^{\nu}$ denoting a smooth domain), one can further reduce the measure support to the Hölder α continuous function in Ω if $2m - \frac{\nu}{2} > \alpha$. Namely, we have a easy proof of the famous Wiener Theorem on sample continuity of certain path integrals in Sobolev Spaces

$$\mathcal{M}^{2m}(\Omega) \subset C^{\alpha}(\Omega) \tag{8-a}$$

The above Wiener Theorem is fundamental in order to construct non-trivial examples of mathematically rigorous euclideans path integrals in spaces R^ν of higher dimensionality, since it is a trivial consequence of the Lebesgue theorem that positive continuous functions $V(x)$ generate functionals integrable in $\{\mathcal{M}^{2m}(\Omega), d_B\mu(\varphi)\}$ of the form below

$$\exp\left\{-\int_\Omega V(\varphi(x))dx\right\} \in L^1(\mathcal{M}^{2m}(\Omega), d_B\mu(\varphi)). \tag{8-b}$$

As a last important remark on Cylindrical Measures in Separable Hilbert Spaces, let us point at to our reader that the support of such above measures is always a σ-compact set in the norm topology of $\mathcal{M}$. In order to see such result let us consider a given dense set of $\mathcal{M}$, namely $\{x_k\}_{k\in I^+}$. Let $\{\delta_k\}_{k\in I^+}$ be a given sequence of positive real numbers with $\delta_k \to 0$. Let $\{\varepsilon_n\}$ another sequence of positive real numbers such that $\sum_{n=1}^\infty \varepsilon_n < \varepsilon$. Now it is straightforward to see that $\mathcal{M} \subset \bigcup_{h=1}^\infty \overline{B(x_k,\delta_k)} \subset \mathcal{M}$ and thus $\limsup \mu\{\bigcup_{k=1}^n \overline{B(x_k,\delta_k)}\} = \mu(\mathcal{M}) = 1$. As a consequence, for each n, there is a k_n, such that $\mu\left(\bigcup_{k=1}^{k_n} \overline{B(x_k,\delta_k)}\right) \geq 1-\varepsilon$.

Now the sets $K_\mu = \bigcap_{n=1}^\infty \left[\bigcup_{k=1}^{k_n} \overline{B(x_k,\delta_k)}\right]$ are closed and totally bounded, so they are compact sets in $\mathcal{M}$ with $\mu(\mathcal{M}) \geq 1-\varepsilon$. Let is now choose $\varepsilon = \frac{1}{n}$ and the associated compact sets $\{K_{n,\mu}\}$. Let us further consider the compact sets $\hat{K}_{n,\mu} = \bigcup_{\ell=1}^n K_{\ell,\mu}$. We have that $\hat{K}_{n,\mu} \subseteq \hat{K}_{n+1,\mu}$, for any n and $\limsup \mu(\hat{K}_{n,\mu}) = 1$. So, $\operatorname{Supp} d\mu = \bigcup_{n=1}^\infty \hat{K}_{n,\mu}$, a σ-compact set of $\mathcal{M}$.

We consider now a enumerable family of cylindrical measures $\{d\mu_n\}$ in $\mathcal{M}$ satisfying the chain inclusion relationship for any $n \in I^+$

$$\operatorname{Supp} d\mu_n \subseteq \operatorname{Supp} d\mu_{n+1}.$$

Now it is straightforward to see that the compact sets $\{\hat{K}_n^{(n)}\}$, where $\operatorname{Supp} d\mu_m = \bigcup_{n=1}^\infty \hat{K}_n^{(m)}$, is such that $\operatorname{Supp}\{d\mu_m\} \subseteq \bigcup_{n=1}^\infty \hat{K}_n^{(n)}$, for any $m \in I^+$.

Let us consider the family of functionals induced by the restriction of this sequence of measures in any compact $\hat{K}_n^{(n)}$. Namely

$$\mu_n \to L_n^{(n)}(f) = \int_{\hat{K}_n^{(n)}} f(x)\cdot d\mu_p(x). \tag{8-c}$$

Here $f \in C_b(\hat{K}_n^{(n)})$. Note that all the above functionals in $\bigcup_{n=1}^\infty C_b(\hat{K}_n^{(n)})$ are bounded by 1. By the Alaoglu-Bourbaki theorem they form a compact set in the weak star topology of

$\left(\bigcup_{n=1}^{\infty} C_b(\hat{K}_n^{(n)})\right)^*$, so there is a sub-sequence (or better the whole sequence) converging to a unique cylindrical measure $\bar{\mu}(x)$. Namely

$$\lim_{n\to\infty} \int_{\mathcal{M}} f(x) d\mu_n(x) = \int_{\mathcal{M}} f(x) d\bar{\mu}(x) \tag{8-d}$$

for any $f \in \bigcup_{n=1}^{\infty} C_b(\hat{K}_n^{(n)})$.